On the homotopy types of the groups of equivariant diffeomorphisms(Dissertation_全文)

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主論文

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§0. Introduction

The purpose of this paper is to study the homotopy type of the group of the equivariant diffeomorphisms of a closed connected smooth G-manifold M, when G is a compact Lie group and the orbit space M/G is homeomorphic to a unit interval [0,1].

Let $\mathrm{Diff}_G^\infty(M)_0$ denote the group of equivariant C^∞ diffeomorphisms of the G-manifold M which are G-isotopic to the identity, endowed with C^∞ topology. If M/G is homeomorphic to [0,1], then M has two or three orbit types G/H, G/K_0 and G/K_1. We can choose the isotropy subgroups H, K_0, K_1 satisfying $H \subset K_0 \cap K_1$. Moreover the G-manifold structure of M is determined by an element η of a factor group N(H)/H, where N(H) is the normalizer of H in G (see §1). Let $\Omega(N(H)/H; (N(H)\cap N(K_0))/H, (N(H)\cap N(\eta K_1\eta^{-1})/H)_0$ denote the connected component of the identity of the space of paths $a: [0,1] \longrightarrow N(H)/H$ satisfying $a(0) \in (N(H)\cap N(K_0))/H$ and $a(1) \in (N(H)\cap N(\eta K_1 \eta^{-1}))/H$.

Theorem. $\mathrm{Diff}_G^\infty(M)_0$ has the same homotopy type as the path space $\Omega(N(H)/H; (N(H)\cap N(K_0))/H, (N(H)\cap N(\eta K_1\eta^{-1}))/H)_0$.

The paper is organized as follows. In §1, we study the G-manifold structure of M and give a differentiable structure of M/G such that the functional structure of M/G is induced from that of M. This differentiable structure is important to study the structure of $\mathrm{Diff}_G^\infty(M)_0$. In §2, we define a group homomorphism P: $\mathrm{Diff}_G^\infty(M)_0 \longrightarrow \mathrm{Diff}^\infty[0,1]_0$ and prove that P is a continuous homomorphism between topological groups. In §3, we prove that there exists a global continuous section of P and Ker P is a deformation retract of $\mathrm{Diff}_G^\infty(M)_0$. In §4, we study the group structure of Ker P. In §5 and §6, we prove our Theorem.

§1. G-manifold structure of M and the functional structure of M/G.

In this paper we assume that all manifolds and all actions are differentiable of class C^∞.

In this section we study the G-manifold structure of M. First we see that it is sufficient for us to consider $\eta=1$ (see Lemma 1.1). Next we give a differentiable structure of M/G such that the functional structure of M/G is induced from that of M (see Lemma 1.2).

Let M be a closed connected smooth G-manifold such that M/G is homeomorphic to [0,1]. We denote this homeomorphism by f. Let $\pi: M \longrightarrow M/G$ be the natural projection. Put $M_0 = (f\circ\pi)^{-1}([0,1/2])$ and $M_1 = (f\circ\pi)^{-1}([1/2,1])$. Let x_i be a point of M with $f(\pi(x_i)) = i$ for $i = 0,1$. Then M_i is a G-invariant closed tubular neighborhood of the orbit $G(x_i)$ (c,f. G. Bredon [3, Chapter VI, §6]). Moreover M is equivariantly diffeomorphic to a union of the G-manifolds M_0 and M_1 such that their boundaries are identified under a G-diffeomorphism $\eta': \partial M_0 \longrightarrow \partial M_1$. Let V_i be a normal vector space of $G(x_i)$ at x_i and K_i be the isotropy subgroup of x_i for $i = 0,1$. Then V_i is a representation space of K_i. From the differentiable slice theorem, M_i is equivariantly diffeomorphic to a smooth G-bundle $G\times_{K_i} D(V_i)$ which is associated to the principal K_i bundle $\pi_i: G \longrightarrow G/K_i$, where $D(V_i)$ is a unit disc in V_i.

Let H be a principal isotropy subgroup of the G-manifold M. We can assume that H is a subgroup of $K_0 \cap K_1$. Let $e_i \in S(V_i)$ be a point such that the isotropy subgroup of e_i is H for $i = 0,1$, where $S(V_i)$ is a unit sphere in V_i. There exists a G-diffeomorphism $h_i: G/H \longrightarrow G\times_{K_i} S(V_i)$ given by $h_i(gH) = [g,e_i]$, $i = 0,1$. Then we have a G-diffeomorphism

$$\eta'' : G/H \xrightarrow{h_0} G\times_{K_0} S(V_0) = \partial M_0 \xrightarrow{\eta'} \partial M_1 = G\times_{K_1} S(V_1) \xrightarrow{h_1^{-1}} G/H.$$

Since any G-map $G/H \longrightarrow G/H$ is given by a right translation of an element of $N(H)/H$, η'' defines an element $\eta \in N(H)/H$.

Put $x_1' = \eta \cdot x_i$. Then the isotropy subgroup K_1' of x_1' is $\eta K_1 \eta^{-1}$. Let V_1' be a normal vector space of the orbit $G(x_1') = G(x_i)$ at x_1'. Put $e_1' = (d\eta)_{x_1}(e_1) \in S(V_1')$. There exists a G-diffeomorphism u: $G \times_{K_1} D(V_1) \longrightarrow G \times_{K_1'} D(V_1')$ given by $u([g,v]) = [g\eta^{-1}, \eta \cdot v]$. Then $(u \circ \eta')([g,e_0]) = u([g\eta, e_1]) = [g, e_1']$ for $[g,v] \in G \times_{K_0} S(V_0)$. Therefore M is equivariantly diffeomorphic to a union of the G-bundles $G \times_{K_0} D(V_0)$ and $G \times_{K_1'} D(V_1')$ such that their boundaries are identified under a G-diffeomorphism $u \circ \eta'$. Now we have:

Lemma 1.1. Let M be a closed connected smooth G-manifold such that M/G is homeomorphic to $[0,1]$. Then M has two or three orbit types G/H, G/K_0 and G/K_1 with $H \subset K_0 \cap K_1$, and there exist representation spaces V_i, $i = 0,1$, of K_i such that M is equivariantly diffeomorphic to a union of G-bundles $G \times_{K_0} D(V_0)$ and $G \times_{K_1} D(V_1)$ with their boundaries identified under a G-diffeomorphism η: $G \times_{K_0} S(V_0) \longrightarrow G \times_{K_1} S(V_1)$. Moreover we may assume that $\eta([g,e_0]) = [g,e_1]$, where e_i is a point of $S(V_i)$ such that the isotropy subgroup of e_i is H for $i = 0,1$.

Hereafter we shall assume that M is a G-manifold as in Lemma 1.1. Let ξ: $[0,1] \longrightarrow R$ be a smooth function such that

$$\xi(r) = r^2 \quad \text{for } 0 \le r \le 1/2,$$

$$\xi'(r) > 0 \quad \text{for } 0 < r \le 1 \text{ and}$$

$$\xi(r) = r - 1/2 \quad \text{for } 7/8 < r \le 1.$$

Let θ: $M = G \times_{K_0} D(V_0) \cup_{\eta} G \times_{K_1} D(V_1) \longrightarrow [0,1]$ be a map given by

$$\theta([g,v]) = \xi(\|v\|) \qquad \text{for } [g,v] \in G \times_{K_0} D(V_0),$$

$$\theta([g,v]) = 1 - \xi(\|v\|) \qquad \text{for } [g,v] \in G \times_{K_1} D(V_1).$$

Since θ is a G-map, there exists a map ϕ: $M/G \longrightarrow [0,1]$ such that $\phi \circ \pi = \theta$. It is easy to see that ϕ is a homeomorphism. We give a differentiable structure of M/G by ϕ.

Lemma 1.2. In the above situation, we have

(1) θ is a smooth map,

(2) there exists a G-diffeomorphism α: $\theta^{-1}((0,1)) \longrightarrow G/H \times (0,1)$ such that $\theta\circ\alpha^{-1}$ is the projection on the second factor, and

(3) $f: M/G \longrightarrow R$ is smooth if and only if $f\circ\pi: M \longrightarrow R$ is smooth.

Proof. (1) Let $\alpha_i: G\times_{K_i}(D(V_i)-0) \longrightarrow G/H \times (0,1]$ be a map given by $\alpha_i([g,re_i]) = (gH,r)$ for $g\in G$ and $r\in(0,1]$ $(i=0,1)$. Then it is easy to see that α_i is a G-diffeomorphism. Since $\alpha_1\circ\eta = \alpha_0$ on $G\times_{K_0} S(V_0)$, the composition β: $\theta^{-1}((0,1)) = G\times_{K_0}(D(V_0)-0)\cup_\eta G\times_{K_1}(D(V_1)-0) \xrightarrow{\alpha_0\cup\alpha_1} G/H \times (0,1]\cup_{1_{G/H}\times 1} G/H\times(0,1] = G/H\times(0,2)$ is a G-diffeomorphism. Note that

$$(\theta\circ\beta^{-1})(gH,r) = \begin{cases} \xi(r) & \text{for } 0<r\leq 1, \\ 1-\xi(2-r) & \text{for } 1\leq r<2. \end{cases}$$

Thus $\theta\circ\beta^{-1}$ is a smooth map, and θ is a smooth map on $\theta^{-1}((0,1))$. From the definition, θ is a smooth map on $\theta^{-1}(r)$ for $r \neq 1/2$. Therefore θ is a smooth map.

(2) Let $\bar\theta$: $(0,2) \longrightarrow (0,1)$ be a smooth map given by

$$\bar\theta(r) = \begin{cases} \xi(r) & \text{for } 0<r\leq 1, \\ 1-\xi(2-r) & \text{for } 1\leq r<2. \end{cases}$$

Since $\bar\theta'(r)>0$ for $0<r<2$, $\bar\theta$ is a diffeomorphism. Let α: $\theta^{-1}((0,1)) \longrightarrow G/H\times(0,1)$ be a G-diffeomorphism given by $\alpha = (1,\bar\theta)\circ\beta$. Then $(\theta\circ\alpha^{-1})(gH,r) = (\theta\circ\beta^{-1})(gH,\bar\theta^{-1}(r)) = r$, and $\theta\circ\alpha^{-1}$ is the projection on the second factor.

(3) Let $f: M/G \longrightarrow R$ be a function such that $f\circ\pi: M\longrightarrow R$ is smooth. We shall prove that $f\circ\phi^{-1}$: $[0,1] \longrightarrow R$ is smooth. Since

$(f\circ\pi\circ\alpha^{-1})(gH,r) = (f\circ\phi^{-1}\circ\theta\circ\alpha^{-1})(gH,r) = (f\circ\phi^{-1})(r)$ for $0<r<1$, $f\circ\phi^{-1}$ is smooth on $(0,1)$. Let i_0: $D_{1/2}(V_0) = \{v\in D(V_0);\ \|v\|\leq 1/2\} \longrightarrow G\times_{K_0} D(V_0)$ be an inclusion given by $i_0(v) = [1,v]$. Note that $(\theta\circ i_0)(v) = \|v\|^2$ for $v\in D_{1/2}(V_0)$. By Corollary 5.3 of G. Bredon [3, Chapter VI, §5], $f\circ\phi^{-1}$ is smooth if and only if $(f\circ\phi^{-1})\circ(\theta\circ i_0)$ is smooth. Since $(f\circ\phi^{-1})\circ(\theta\circ i_0) = f\circ\pi\circ i_0$, which is smooth, then $f\circ\phi^{-1}$ is smooth on $[0,1/4]$. Similarly we can prove that $f\circ\phi^{-1}$ is smooth on $[3/4,1]$. Since $(f\circ\phi^{-1})(r) = (f\circ\phi^{-1}\circ\theta\circ\alpha^{-1})(1H,r) = (f\circ\pi\circ\alpha^{-1})(1H,r)$ for $0<r<1$, $f\circ\phi^{-1}$ is smooth on $(0,1)$. This completes the proof of Lemma 1.2.

Remark 1.3. Lemma 1.2 is essentially proved by G. Bredon [3, Chapter VI, §5], and (3) implies that the functional structure of M/G is induced from that of M.

§2. On the group homomorphism P.

In this section we shall define a group homomorphism $P: \mathrm{Diff}_G^\infty(M)_0 \longrightarrow \mathrm{Diff}^\infty[0,1]_0$, and we shall prove P is continuous.

We shall identify the orbit space M/G with $[0,1]$ by the homeomorphism ϕ in §1, therefore the projection $\pi: M \longrightarrow M/G$ is identified with the smooth map $\theta: M \longrightarrow [0,1]$. Let $h: M \longrightarrow M$ be a G-diffeomorphism of M which is G-isotopic to the identity 1_M, and let $f: [0,1] \longrightarrow [0,1]$ be the orbit map of h. Since $f\circ\pi = \pi\circ h$ is a smooth map, f is a smooth map by Lemma 1.2 (3). Similarly the inverse map f^{-1} of f is smooth, and f is a diffeomorphism. Then there exists an abstract group homomorphism $P: \mathrm{Diff}_G^\infty(M)_0 \longrightarrow \mathrm{Diff}^\infty[0,1]$ which is given by $P(h) = f$, where $\mathrm{Diff}^\infty[0,1]$ is the group of C^∞ diffeomorphism of $[0,1]$, endowed with C^∞ topology.

Proposition 2.1. $P: \mathrm{Diff}_G^\infty(M)_0 \longrightarrow \mathrm{Diff}^\infty[0,1]$ is a continuous homomorphism of topological groups.

Let $C^\infty(M_1,M_2)$ denote the set of all smooth maps from a compact smooth manifold M_1 to a smooth manifold M_2, endowed with C^∞ topology. Before the proof of Proposition 2.1, we begin with some lemmas.

Lemma 2.2. Let M_i be a compact smooth manifold and N_i be a smooth manifold for $i = 1,2$. Then we have

(1) Let $\phi: N_1 \longrightarrow N_2$ be a smooth map, and let $\phi_*: C^\infty(M_1,N_1) \longrightarrow C^\infty(M_1,N_2)$ be a map which is given by $\phi_*(f) = \phi\circ f$. Then ϕ_* is continuous.

(2) Let $\phi: M_1 \longrightarrow M_2$ be a smooth map, and let $\phi^*: C^\infty(M_2,N_1) \longrightarrow C^\infty(M_1,N_1)$ be a map which is given by $\phi^*(f) = f\circ\phi$. Then ϕ^* is continuous.

(3) Let $\phi: M_1 \longrightarrow N_2$ be a smooth map and let $\phi_\# : C^\infty(M_1,N_1) \longrightarrow C^\infty(M_1,N_1\times N_2)$ be a map which is given by $\phi_\#(f) = (f,\phi)$. Then $\phi_\#$ is continuous.

(4) Let $\phi\colon M_2 \longrightarrow N_2$ be a smooth map and let $\phi_!\colon C^\infty(M_1,N_1) \longrightarrow C^\infty(M_1\times M_2, N_1\times N_2)$ be a map given by $\phi_!(f) = f\times\phi$. Then $\phi_!$ is continuous.

(5) Let $\kappa\colon C^\infty(M_1,N_1)\times C^\infty(M_1,N_2) \longrightarrow C^\infty(M_1,N_1\times N_2)$ be a map given by $\kappa(f,g)(x) = (f(x),g(x))$ for $x\in M_1$. Then κ is continuous.

(6) Let L be a smooth manifold. Let comp: $C^\infty(M_1,N_1)\times C^\infty(N_1,L) \longrightarrow C^\infty(M_1,L)$ be a map given by $\mathrm{comp}(f,g) = g\circ f$. Then comp is continuous.

Proof. (1) and (2) are proved by R. Abraham [2, Theorem 11.2 and 11.3]. It is easy to see (3), (4) and (5). From J. Cerf [4, Chapter I, §4, Proposition 5], (6) follows.

Lemma 2.3. Let X be a topological space. Let M be a compact smooth manifold and N be a smooth manifold. Choose an open covering $\{U_i\}$ of M such that each closure $\bar{U}_i$ of U_i is a regular submanifold of M which is contained in a coordinate neighborhood of M. Then a map $\Psi\colon X \longrightarrow C^\infty(M,N)$ is continuous if and only if each composition $\Psi_i\colon X \xrightarrow{\Psi} C^\infty(M,N) \xrightarrow{j_i^*} C^\infty(\bar{U}_i,N)$ is continuous for each i, where $j_i\colon \bar{U}_i \hookrightarrow M$ is an inclusion.

Proof. From Lemma 2.2 (2), if Ψ is continuous, then Ψ_i is continuous for each i. We can choose $\{U_i\}$ as a coordinate covering of M. Let $\{V_\lambda\}$ be a coordinate covering of N. Let $f\in C^\infty(M,N)$ and $K\subset U_i$ be a compact subset such that $f(K)\subset V_\lambda$ for some λ. $N^r(f,U_i,V_\lambda,K,\varepsilon)$ $(r=0,1,2,\ldots,\ 0<\varepsilon\leq\infty)$ denotes the set of C^r maps $g\colon M\longrightarrow N$ such that $g(K)\subset V_\lambda$ and $\|D^kf(x)-D^kg(x)\|<\varepsilon$ for any $x\in K$, $k=0,1,2,\ldots,r$. Then the C^∞ topology on $C^\infty(M,N)$ is generated by these sets $N^r(f, U_i, V_\lambda, K,\varepsilon)$ (see M. Hirsch [6, Chapter 2, §1]).

Let $x\in X$ and let $f=\Psi(x)$. For any open neighborhood W of f, there exist above sets $N_k = N^{r_k}(f,U_{i_k},V_{\lambda_k},K_k,\varepsilon_k)$, $k=1,2,\ldots,n$, such that $\bigcap_{k=1}^n N_k\subset W$. Note that $j_{i_k}^*\colon C^\infty(M,N)\longrightarrow C^\infty(\bar{U}_{i_k},N)$ is an open map and $(j_{i_k}^*)^{-1}(j_{i_k}^*(N_k)) = N_k$. Since Ψ_{i_k} is continuous, $\Psi^{-1}(N_k) = \Psi_{i_k}^{-1}(j_{i_k}^*(N_k))$ is an open neighborhood of x in X, for each k. Then $\bigcap_{k=1}^n$

$\Psi^{-1}(N_k)$ is an open neighborhood of x in X. Since $\Psi(\bigcap_{k=1}^{n} \Psi^{-1}(N_k)) \subset \bigcap_{k=1}^{n} N_k \subset W$, Ψ is continuous at x. This completes the proof of Lemma 2.3.

Remark. Lemma 2.2 and Lemma 2.3 are hold in the cases of manifolds with corneres.

Let $C_e^\infty([-1/2,1/2], R)$ denote the set of all smooth functions $f: [-1/2,1/2] \longrightarrow R$ satistying $f(-x) = f(x)$, endowed with C^∞ topology. Let $T: C_e^\infty([-1/2,1/2], R) \longrightarrow C^\infty([0,1/4], R)$ denote a map defined by $T(f)(x) = f(\sqrt{x})$. Then we have

Lemma 2.4. The above map T is well defined and continuous.

Proof. Put $f = T(\hat{f})$ for each $\hat{f} \in C_e^\infty([-1/2,1/2], R)$. Since $\hat{f}$ is a C^∞ even function, we have the Taylor expansion

$$\hat{f}(x) = \hat{f}(0) + (\hat{f}''(0)/2)x^2 + \dots + (\hat{f}^{(2n-2)}(0)/(2n-2)!)x^{2n-2} + (\int_0^1 ((1-t)^{2n-1}/(2n-1)!)\hat{f}^{(2n)}(tx)\, dt)x^{2n}$$

for $-1/2 \le x \le 1/2$, $n=1,2,\dots$. Thus we have

$$f(x) = \hat{f}(0) + (\hat{f}''(0)/2)x + \dots + (\hat{f}^{(2n-2)}(0)/(2n-2)!)x^{n-1} + (\int_0^1 ((1-t)^{2n-1}/(2n-1)!)\hat{f}^{(2n)}(t\sqrt{x})\, dt)x^n$$

for $0 \le x \le 1/4$. By the composite mapping formula, we can compute the n-th derivative

$$D^n(f^{(2n)}(t\sqrt{x})x^n) = \Sigma_{p=0}^{n} \Sigma_{q=0}^{p} \Sigma_{\substack{i_1+\dots i_q = p \\ i_1>0,\dots,\, i_q>0}} B(p,i_1,\dots,i_q)\hat{f}^{(2n+q)}(t\sqrt{x})x^{q/2}t^q,$$

where $B(p,i_1,\dots,i_q)$ is a real number. Put $f_i = T(\hat{f}_i)$ for $\hat{f}_i \in C_e^\infty([-1/2,1/2], R)$ $(i=1,2)$. Then there exists a positive number A_n such that

$$\sup_{0\le x\le 1/4} |D^n f_1(x) - D^n f_2(x)| \le A_n \cdot \max_{0\le q\le 3n} (\sup_{-1/2\le x\le 1/2} |D^q\hat{f}_1(x) - D^q\hat{f}_2(x)|)$$

for each $n = 1, 2, \ldots$. Note that

$$\sup_{0\le x\le 1/4} |f_1(x) - f_2(x)| = \sup_{-1/2\le x\le 1/2} |\hat{f}_1(x) - \hat{f}_2(x)|.$$

Therefore T is a continuous map, and this completes the proof of Lemma 2.4.

Proof of Proposition 2.1. Let J denote a closed interval $[0,1/4]$, $[1/5,4/5]$ or $[3/4,1]$. By Lemma 2.3, it is sufficient to show that the composition P_J: $\mathrm{Diff}_G^\infty(M)_0 \xrightarrow{P} \mathrm{Diff}^\infty[0,1] \xrightarrow{j^*} C^\infty(J,[0,1])$ is continuous, where j: $J \hookrightarrow [0,1]$ is an inclusion map.

We shall first consider the case $J = [0,1/4]$. Let $\imath: [-1/2,1/2] \longrightarrow [0,1/4]$ be a map given by $\imath(x) = x^2$. Let $\hat{\imath}$: $[-1/2,1/2] \longrightarrow G\times_{K_0} D(V_0) \hookrightarrow M$ be a map given by $\hat{\imath}(r) = [1, re_0]$, where e_0 is a point of $S(V_0)$ as in §1. Then $\pi\circ\hat{\imath} = \imath$. Let $\hat{P}_J$ denote the composition $\mathrm{Diff}_G^\infty(M)_0 \xrightarrow{\hat{\imath}^*} C^\infty([-1/2,1/2], M) \xrightarrow{\pi_*} C^\infty([-1/2,1/2], [0,1])$. Then $\hat{P}_J(h) = \pi\circ h\circ\hat{\imath} = P(h)\circ\imath = \imath^*P(h)$ for $h \in \mathrm{Diff}_G^\infty(M)_0$, and the image of $\hat{P}_J$ is contained in $C_e^\infty([-1/2,1/2], R)$. Note that $P_J = T\circ\hat{P}_J$. Combining Lemma 2.2 and Lemma 2.4, P_J is continuous.

Next consider the case $J = [1/5,4/5]$. By Lemma 1.2, there is a G-diffeomorphism α: $\pi^{-1}([1,5,4/5]) \longrightarrow G/H \times [1/5,4/5]$. Let i: $\pi^{-1}([1/5,4/5]) \hookrightarrow M$ be the inclusion map and let k: $[1/5,4/5] \longrightarrow G/H \times [1/5,4/5]$ be a map given by $k(r) = (1H, r)$. Then P_J is the composition

$$\mathrm{Diff}_G^\infty(M)_0 \xrightarrow{(i\circ\alpha^{-1}\circ k)^*} C^\infty([1/5,4/5], M) \xrightarrow{\pi_*} C^\infty([1/5,4/5],[0,1])$$

which is continuous by Lemma 2.2.

We can prove that P_J is continuous in the case $J = [3/4,1]$ similarly as in the case $J = [0,1/4]$, and this completes the proof of Proposition 2.1.

§3. A continuous global section of P.

In §2 we have proved that $P: \mathrm{Diff}_G^\infty(M)_0 \longrightarrow \mathrm{Diff}^\infty[0,1]$ is continuous. Thus the image of P is contained in the connected component $\mathrm{Diff}^\infty[0,1]_0$ of the identity. In this section we shall construct a continuous global section of $P: \mathrm{Diff}_G^\infty(M)_0 \longrightarrow \mathrm{Diff}^\infty[0,1]_0$.

Let f be an element of $\mathrm{Diff}^\infty[0,1]_0$. We shall define a map $\Psi(f): M \longrightarrow M$ as follows: $\Psi(f)$ is defined on $\pi^{-1}((0,1))$ by the composition $\pi^{-1}((0,1)) \xrightarrow{\alpha} G/H \times (0,1) \xrightarrow{(1,f)} G/H \times (0,1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0,1))$, and $\Psi(f) = 1$ on $\pi^{-1}(0) \cup \pi^{-1}(1)$.

Proposition 3.1. $\Psi(f)$ is a G-diffeomorphism of M.

In order to prove Proposition 3.1, we need the following lemma and notations.

Lemma 3.2. Let $\Psi_1: \mathrm{Diff}^\infty[0,1]_0 \longrightarrow \mathrm{Diff}^\infty(D^n)$ be a map defined by

$$\Psi_1(f)(v) = \begin{cases} (\sqrt{f(\|v\|^2)}/\|v\|)v & \text{for } v \neq 0, \\ 0 & \text{for } v = 0, \end{cases}$$

where D^n denotes an n-dimensional unit disc. Then Ψ_1 is a well defined and continuous map.

Notations 3.3. For $i = 0,1$, we shall use the following notations:

$\pi_i: G \longrightarrow G/K_i$ the natural projection,

U_i an open disc neighborhood of $1K_i$ in G/K_i,

$\sigma_i: U_i \longrightarrow G$ a smooth local cross section of π_i,

$\sigma_{i,a}: aU_i \longrightarrow G$ $(a \in G)$ a smooth local cross section of π_i defined by $\sigma_{i,a}(x) = a \cdot \sigma_i(a^{-1}x)$.

Put $M_i = G \times_{K_i} D(V_i)$ and $M_i(r) = G \times_{K_i} D_r(V_i)$, where $D_r(V_i)$ is a closed r-disc in V_i $(0 < r \leq 1)$.

$p_i: M_i \longrightarrow G/K_i$, $p_{i,r}: M_i(r) \longrightarrow G/K_i$ the bundle projections,

$\phi_{i,a}: p_i^{-1}(aU_i) \longrightarrow U_i \times D(V_i)$ $(a \in G)$ a chart of p_i over aU_i defined by $\phi_{i,a}([g,v]) = (a^{-1}\pi_i(g), ((\sigma_{i,a}\circ\pi_i)(g))^{-1}g\cdot v)$,

$\pi_2: G \longrightarrow G/H$ the natural projection,

U_2 an open disc neighborhood of $1H$ in G/H,

$\sigma_2: U_2 \longrightarrow G$ a smooth local cross section of π_2.

Proof of Proposition 3.1. Put $h = \Psi(f)$. We shall first prove that h is smooth on $\pi^{-1}(0)$. Since $f(0)=0$, there exists a real number ε such that $0<\varepsilon\leq 1/2$ and $f(\varepsilon^2)\leq 1/4$. Then $h(\pi^{-1}([0,\varepsilon^2]) \subset \pi^{-1}([0,1/4])$, and $h(M_0(\varepsilon)) \subset M_0(1/2)$. For $[g,re_0] \in G\times_{K_0} D_\varepsilon(V_0-0)$ $(0<r\leq\varepsilon)$, $h([g,re_0]) = (\alpha^{-1}\circ(1,f)\circ\alpha)([g,re_0]) = (\alpha^{-1}\circ(1,f))(gH,r^2) = \alpha^{-1}(gH,f(r^2))$ $=[g,\sqrt{f(r^2)}e_0]$. Then, for $[g,v] \in G\times_{K_0} D_\varepsilon(V_0-0)$, $h([g,v]) = [g,\sqrt{f(\|v\|^2)}/\|v\|\, v] = [g,\Psi_1(f)(v)]$. Since $h([g,0]) = [g,0]$, $h([g,v]) = [g,\Psi_1(f)(v)]$ for any $[g,v]\in M_0(\varepsilon)$. Then the composition

$$\begin{aligned}\tilde{h}\colon U_0\times D_\varepsilon(V_0) &\xrightarrow{(\phi_{0,a})^{-1}} p_{0,\varepsilon}^{-1}(aU_0)\\ &\xrightarrow{\ h\ } p_{0,1/2}^{-1}(aU_0)\\ &\xrightarrow{\phi_{0,a}} U_0\times D_{1/2}(V_0)\end{aligned}$$

is given by $\tilde{h}(x,v) = (x,\Psi_1(f)(v))$ for $a\in G$. Since $\Psi_1(f)$ is a smooth map by Lemma 3.2, h is smooth on $\pi^{-1}(0)$. Similarly we can prove that h is smooth on $\pi^{-1}(1)$. Since h is smooth on $\pi^{-1}((0,1))$ by the definition, h is a smooth map. Since $h^{-1} = \Psi(f^{-1})$, h^{-1} is also a smooth map. Thus h is a G-diffeomorphism of M, and this completes the proof of Proposition 3.1.

In order to prove Lemma 3.2, we need the following assertion.

Assertion 3.4. Let $\Phi\colon \mathrm{Diff}^\infty[0,1]_0 \longrightarrow C^\infty([0,1],R)$ be a map given by

$$\Phi(f)(x) = \begin{cases}\sqrt{f(x)/x} & \text{for } x \neq 0,\\ \sqrt{f'(0)} & \text{for } x = 0.\end{cases}$$

Then Φ is a well defined continuous map.

Proof. For $f\in \mathrm{Diff}^\infty[0,1]_0$, we have the Taylor expansion

$$f(x) = f'(0)x + x^2\int_0^1 (1-t)f''(tx)\,dt \qquad \text{for } 0\le x\le 1.$$

Put $F(x) = f'(0) + x\int_0^1 (1-t)f''(tx)\,dt$ for $0\le x\le 1$. Then $\Phi(f) = \sqrt{F}$. Note that $F(x)>0$ for $0\le x\le 1$. It is easy to see that Φ is continuous.

Proof of Lemma 3.2. Let $N\colon D^n \longrightarrow [0,1]$ be a map given by $N(v) = \|v\|^2$. Let $i\colon D^n \hookrightarrow R^n$ be the inclusion and let $\mu\colon R\times R^n \longrightarrow R^n$ be the scalar multiplication. Since $\Psi_1(f)(v) = \Phi(f)(\|v\|^2)v$, $\Psi_1(f)$ is a smooth map by Assertion 3.4. Since $\Psi_1(f^{-1}) = \Psi_1(f)^{-1}$, $\Psi_1(f)^{-1}$ is also a smooth map. Thus $\Psi_1(f)$ is a diffeomorphism of D^n. Note that Ψ_1 is the composition $\mathrm{Diff}^\infty[0,1]_0 \xrightarrow{\Phi} C^\infty([0,1],R) \xrightarrow{N^*} C^\infty(D^n,R) \xrightarrow{i_\#} C^\infty(D^n,R\times R^n) \xrightarrow{\mu_*} C^\infty(D^n,R^n)$. Combining Assertion 3.4 and Lemma 2.2, Ψ is continuous. This completes the proof of Lemma 3.2.

Proposition 3.5. $\Psi\colon \mathrm{Diff}^\infty[0,1]_0 \longrightarrow \mathrm{Diff}^\infty_G(M)$ is continuous.

Proof. Let $B_i \subset U_i$ be a closed disc neighborhood of $1K_i$ in G/K_i for $i=0,1$. Let $B_2 \subset U_2$ be a closed disc neighborhood of $1H$ in G/H. We can choose $\{\mathrm{int}(p^{-1}_{0,\varepsilon}(aB_0)),\ \mathrm{int}(p^{-1}_{1,\varepsilon}(aB_1)),\ \mathrm{int}(\alpha^{-1}(aB_2\times[\varepsilon/2,1-\varepsilon/2]))\ ;\ a\in G\}$ as an open covering of M for $0<\varepsilon<1/2$. Put $W = \{f\in \mathrm{Diff}^\infty[0,1]_0\ ;\ f([0,\varepsilon^2]) \subset [0,1/4),\ f([1-\varepsilon^2,1])\subset(3/4,1]\}$. Then W is an open neighborhood of the identity in $\mathrm{Diff}^\infty[0,1]_0$. Since Ψ is a homomorphism as an abstract group, it is enough to show that Ψ is continuous on W. Let C denote one of the sets $p^{-1}_{0,\varepsilon}(aB_0)$, $p^{-1}_{1,\varepsilon}(aB_1)$ or $\alpha^{-1}(aB_2\times[\varepsilon/2,1-\varepsilon/2])$ for $a \in G$. If we can prove that the composition

$$\Psi_C\colon W \xrightarrow{\Psi} \mathrm{Diff}^\infty_G(M)_0 \xrightarrow{i^*} C^\infty(C,M)$$

is continuous for each C, then Ψ is continuous on W by Lemma 2.3, where $i\colon C\hookrightarrow M$ is an inclusion map.

First consider in the case $C = p^{-1}_{0,\varepsilon}(aB_0)$. $\Psi(f)(C)$ is contained in $p^{-1}_{0,1/2}(aU_0)$ for each $f\in W$. Note that $\Psi(f)([g,v]) = [g,\Psi_1(f)(v)]$ for $[g,v]\in C$ and $(\phi_{0,a}\circ\Psi(f)\circ\phi^{-1}_{0,a})(x,v) = (x,\Psi_1(f)(v))$ for $(x,v)\in B_0\times D_\varepsilon(V_0)$. Thus Ψ_C is given by the composition

$$W \xrightarrow{\Psi_1} C^\infty(D_\varepsilon(V_0),\ D(V_0))$$
$$\xrightarrow{j_!} C^\infty(B_0 \times D_\varepsilon(V_0),\ U_0 \times D(V_0))$$
$$\xrightarrow{\phi_{0,a}^{\ *}} C^\infty(C,\ U_0 \times D(V_0))$$
$$\xrightarrow{(k\circ\phi_{0,a}^{-1})_*} C^\infty(C,M),$$

where $j\colon B_0 \hookrightarrow U_0$ and $k\colon p_0^{-1}(aU_0) \hookrightarrow M$ are inclusions. Combining Lemma 3.2 and Lemma 2.2, Ψ_C is continuous.

Now consider the case $C = \alpha^{-1}(B_0 \times [\varepsilon/2, 1-\varepsilon/2])$. Ψ_C is given by the composition

$$W \xrightarrow{\iota^*} C^\infty([\varepsilon/2, 1-\varepsilon/2],\ (0,1))$$
$$\xrightarrow{j_!} C^\infty(B_0 \times [\varepsilon/2, 1-\varepsilon/2],\ G/H \times (0,1))$$
$$\xrightarrow{\alpha^*} C^\infty(C,\ G/H \times (0,1))$$
$$\xrightarrow{(k\circ\alpha^{-1})_*} C^\infty(C,M),$$

where $\iota\colon [\varepsilon/2, 1-\varepsilon/2] \hookrightarrow [0,1]$, $j\colon B_0 \hookrightarrow G/H$ and $k\colon \pi^{-1}((0,1)) \hookrightarrow M$ are inclusion maps. By Lemma 2.2, Ψ_C is continuous.

We can prove that Ψ_C is continuous in the case $C = p_{1,\varepsilon}^{-1}(aB_1)$ similarly as in the case $C = p_{0,\varepsilon}^{-1}(aB_0)$, and this completes the proof of Proposition 3.5.

By Proposition 3.5, $P\colon \mathrm{Diff}_G^\infty(M)_0 \longrightarrow \mathrm{Diff}^\infty[0,1]_0$ is a globally trivial fibration. Then we have

Corollary 3.6. $\mathrm{Diff}_G^\infty(M)_0$ is homeomorphic to $\mathrm{Diff}^\infty[0,1]_0 \times \operatorname{Ker} P$.

§4. On the group Ker P.

In this section we shall define a group homomorphism L: Ker P $\longrightarrow Q$, where Q is a subgroup of $C^{\infty}([0,1], N(H)/H)$, and we shall prove that L is a group monomorphism between topological groups (see Lemma 4.5 and Proposition 4.6).

Let h be an element of Ker P. Let $\hat{h}$ be the composition

$$G/H \times (0,1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0,1)) \xrightarrow{h} \pi^{-1}((0,1)) \xrightarrow{\alpha} G/H \times (0,1).$$

Then $\hat{h}$ is a level preserving G-diffeomorphism. Let $a\colon (0,1) \longrightarrow N(H)/H$ be a smooth map satisfying $h(gH,r) = (ga(r),r)$ for $(gH,r) \in G/H \times (0,1)$.

Proposition 4.1. With the above notations, there exists a smooth map $\bar{a}\colon [0,1] \longrightarrow N(H)/H$ such that

(1) $\bar{a} = a$ on $(0,1)$,

(2) $\bar{a}(i) \in (N(H) \cap N(K_i))/H$ for $i = 0,1$.

To prove Proposition 3.1, we need the following lemma.

Lemma 4.2. Let G be a compact Lie group. Let K and N be closed subgroups of G. Let $\pi\colon G \longrightarrow G/K$ be the natural projection. Then there exists a smooth local section σ of π, which is defined on an open neighborhood U of 1K, such that $\sigma(1K) = 1$ and $\sigma(x) \in N$ for $x \in \pi(N) \cap U$.

Proof. Let $\pi_1\colon N \longrightarrow N/(N \cap K)$ be a natural projection. Let $i\colon N \hookrightarrow G$ be the inclusion and let $I\colon N/(N \cap K) \longrightarrow G/K$ be a map satisfying $\pi \circ i = I \circ \pi_1$. Since $I(N/(N \cap K)) = \pi(N)$ is an orbit of the natural action $N \times G/K \longrightarrow G/K$, I is an imbedding. Let U be a disc neighborhood around $\pi(1)$ in G/K and let U_1 be a disc neighborhood around $\pi_1(1)$ in $N/(N \cap K)$. Since I is an imbedding, we can assume $I(U_1) = U \cap I(N/(N \cap K)) = U \cap \pi(N)$. Let $\sigma_1\colon U_1 \longrightarrow N$ be a smooth local section of π_1 satisfying $\sigma_1(\pi(1)) = 1$. Then $\sigma_1 \circ I^{-1}$ is a smooth section defined on $I(U_1)$. We can extend $\sigma_1 \circ I^{-1}$ to a smooth local section defined on U. Then $\sigma(\pi(1)) = 1$ and $\sigma(U \cap \pi(N)) \subset N$. This completes the proof of Lemma 4.2.

Lemma 4.3. Let G be a compact connected Lie group. Let V be a representation of G such that G acts transitively and effectively on a unit sphere $S(V)$ of V. Let H be the isotropy subgroup of a point of $S(V)$. Then we have the following list:

G	$SO(n)$ $(n \geq 3)$	$SU(n)$ $(n \geq 2)$	$U(n)$ $(n \geq 1)$	$Sp(n)$ $(n \geq 1)$	$Sp(n) \times_{Z_2} S^3$ $(n \geq 1)$
H	$SO(n-1)$	$SU(n-1)$	$U(n-1)$	$Sp(n-1)$	H_1
$N(H)/H$	Z_2	$U(1)$	$U(1)$	$Sp(1)$	Z_2

$Sp(n) \times_{Z_2} S^1$ $(n \geq 1)$	G_2	$Spin(7)$	$Spin(9)$
H_2	$SU(3)$	G_2	$Spin(7)$
S^1	Z_2	Z_2	Z_2

where $H_1 = \{[(q,A),q^{-1}] \in Sp(n) \times_{Z_2} S^3;\ (q,A) \in Sp(1) \times Sp(n-1) \subset Sp(n)\}$ and $H_2 = \{[(z,A),z^{-1}] \in Sp(n) \times_{Z_2} S^1;\ (z,A) \in S^1 \times Sp(n-1) \subset Sp(n)\}$.

Proof. It is known that G and H are the above Lie groups (c.f. W. C. Hsiang and W. Y. Hsiang [7, §1]). We can determine the Lie group $N(H)/H$ by an immediate calculation except for $G = G_2$, $Spin(7)$, $Spin(9)$. For the cases $G = G_2$, $Spin(7)$, $Spin(9)$, we can determine $N(H)/H$ by using I. Yokota's definitions of these Lie groups in [9, Chapter 5].

Lemma 4.4. (1) Let $F\colon [-1,1] \longrightarrow R$ be a smooth function such that $F(0) = 0$. Put $f(x) = F(x)/x$ for $x \neq 0$ and $f(x) = F'(0)$ for $x = 0$. Then $f\colon [-1,1] \longrightarrow R$ is a well defined smooth function.

(2) Put $C_0^\infty([-1,1],R) = \{F \in C^\infty([-1,1],R);\ F(0) = 0\}$, ~~endowed with~~

endowed with C^∞ topology. Let Φ: $C_0^\infty([-1,1],R) \longrightarrow C^\infty([-1,1],R)$ be a map given by $\Phi(F)(x) = f(x)$. Then Φ is continuous.

Proof. For $F \in C_0^\infty([-1,1],R)$, we have $\Phi(F)(x) = f(x) = F'(0) + x\int_0^1 (1-t)F''(tx)\,dt$. Then the n-th derivative $f^{(n)}(x) = x\int_0^1 (1-t)\cdot t^n F^{(n+2)}(tx)\,dt + n\int_0^1 (1-t)t^{n-1}F^{(n+1)}(tx)\,dt$. Thus there exists a positive number A such that $\|\Phi(F)\|_n \leq A\|F\|_{n+2}$, and Lemma 4.4 follows.

Proof of Proposition 4.1. Let ε $(0<\varepsilon\leq 1/2)$ be a real number. Let W_i and U_i be open neighborhoods of $1K_i$, satisfying $\bar{W}_i \subset U_i$. for $i = 0,1$. Put $O = \{h\in \operatorname{Ker} P\ ;\ h(p_{i,\varepsilon}^{-1}(\bar{W}_i)) \subset p_{i,\varepsilon}^{-1}(U_i) \text{ for } i = 0,1\}$. Then O is an open neighborhood of the identity in Ker P. By Corollary 3.6, Ker P is connected, and O generates the topological group Ker P. Thus we can assume $h\in O$.

Let $\tilde{h}$ be the composition

$$W_0 \times D_\varepsilon(V_0) \xrightarrow{(\phi_{0,1})^{-1}} p_{0,\varepsilon}^{-1}(W_0) \xrightarrow{h} p_{0,\varepsilon}^{-1}(U_0) \xrightarrow{\phi_{0,1}} U_0 \times D_\varepsilon(V_0).$$

Let ρ_1: $U_0 \times D_\varepsilon(V_0) \longrightarrow U_0$ and ρ_2: $U_0 \times D_\varepsilon(V_0) \longrightarrow D_\varepsilon(V_0)$ be projections on the first factor and the second factor, respectively. Let i: $[-\varepsilon,\varepsilon] \longrightarrow W_0 \times D_\varepsilon(V_0)$ be an imbedding given by $i(r) = (1K_0, re_0)$. Then the compositions $\tilde{h}_1 = \rho_1\circ\tilde{h}\circ i$: $[-\varepsilon,\varepsilon] \longrightarrow U_0$ and $\tilde{h}_2 = \rho_2\circ\tilde{h}\circ i$: $[-\varepsilon,\varepsilon] \longrightarrow D_\varepsilon(V_0)$ are smooth maps. Let $\bar{\pi}_0$: $G/H \longrightarrow G/K_0$ be the natural projection. Note that

$$\begin{aligned}(\alpha\circ h\circ\phi_{0,1}^{-1})(1K_0,re_0) &= (\alpha\circ h)([1,re_0]) \\ &= (\hat{h}\circ\alpha)([1,re_0]) \\ &= \hat{h}(1H,r^2) \\ &= (a(r^2),r^2) \qquad \text{for } |r|\leq\varepsilon,\ r\neq 0.\end{aligned}$$

Then

$$\begin{aligned}\tilde{h}(1K_0,re_0) &= (\phi_{0,1}\circ\alpha^{-1})(a(r^2),r^2) \\ &= (\bar{\pi}_0(a(r^2)),\ (\sigma_0\circ\bar{\pi}_0)(a(r^2))^{-1}\cdot a(r^2)\cdot re_0),\end{aligned}$$

and

$$\tilde{h}_1(r) = \bar{\pi}_0(a(r^2)),$$
$$\tilde{h}_2(r) = (\sigma_0 \circ \bar{\pi}_0)(a(r^2))^{-1} \cdot a(r^2) \cdot re_0,$$

for $|r| \leq \varepsilon$, $r \neq 0$.

Here we can assume that $\sigma_0(1K_0) = 1$ and $\sigma_0(\pi_0(N(H)) \cap U_0) \subset N(H)$ by Lemma 4.2. Let $b: [-\varepsilon, \varepsilon] \longrightarrow G$ be a smooth map given by $b(r) = \sigma_0(\tilde{h}_1(r))$. Then $b(r) = \sigma_0(\bar{\pi}_0(a(r^2)) \in \sigma_0(\pi_0(N(H)) \cap U_0)$, and $b(r) \in N(H)$ for $r \neq 0$. Since b is a smooth map, $b(r) \in N(H)$ for $r = 0$. For $[1,0] \in \pi^{-1}(0)$, we have $h([1,0]) = (h \circ \phi_{0,1}^{-1})(1K_0, 0) = (h \circ \phi_{0,1}^{-1})(i(0)) = \phi_{0,1}^{-1}(\tilde{h}_1(0), 0) = [b(0), 0]$. Note that p_0 is a G-diffeomorphism on the zero section of p_0 and $h(\pi^{-1}(0)) = \pi^{-1}(0)$. Then the composition $p_0 \circ h \circ p_0^{-1}: G/K_0 \longrightarrow G/K_0$ is a G-diffeomorphism, and $(p_0 \circ h \circ p_0^{-1})(1K_0) = (p_0 \circ h)([1,0]) = p_0([b(0), 0]) = b(0)K_0$. Thus $b(0) \in N(K_0)$, and $b(0) \in N(H) \cap N(K_0)$.

Put $J = [-\varepsilon, 0) \cup (0, \varepsilon]$. Let $c: J \longrightarrow N(H)/H$ be a smooth map given by $c(r) = b(r)^{-1} \cdot a(r^2)$. Since $\bar{\pi}_0(c(r)) = \bar{\pi}_0(\sigma_0(\bar{\pi}_0(a(r^2))^{-1} \cdot a(r^2)) = 1K_0$, then $c(r) \in K_0/H$. Thus $c(r) \in N(H, K_0)/H$ for $r \in J$. Since $\operatorname{Ker} P$ is connected, the maps a, b and c are homotopic to the constant maps. Note that the identity component $(N(H,K_0)/H)^0$ of $N(H,K_0)/H$ is contained in $(N(H,K_0) \cap K_0^0) \cdot H/H$, and there exists an isomorphism $(N(H,K_0) \cap K_0^0) \cdot H/H \simeq (N(H,K_0) \cap K_0^0)/(H \cap K_0^0)$ as a Lie group, where K_0^0 is the identity component of K_0. Then there exists a smooth map $\hat{c}: J \xrightarrow{c} (N(H,K_0)/H)^0 \hookrightarrow (N(H,K_0) \cap K_0^0) \cdot H/H \simeq (N(H,K_0) \cap K_0^0)/(H \cap K_0^0) \hookrightarrow N(H \cap K_0^0, K_0^0)/(H \cap K_0^0)$. Now we shall prove that $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$, and so is c.

Note that K_0 acts transitively on the unit sphere $S(V_0)$ of V_0. If $\dim S(V_0) = 0$, then $K_0/H = Z_2$ and $N(H,K_0)/H = Z_2$. In this case $\hat{c}$ is a trivial map, and it is clear that $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$. Now we assume $\dim S(V_0) > 0$. Since $S(V_0)$ is connected, K_0^0 acts transitively on $S(V_0)$ and $K_0^0/(K_0^0 \cap H)$ is diffeomorphic to $S(V_0)$.

Put $D = \bigcap_{g \in K_0^0} g(K_0^0 \cap H)g^{-1}$ which is the kernel of the action $K_0^0 \times S(V_0) \longrightarrow S(V_0)$. Put $\bar{K}_0 = K_0^0/D$ and $\bar{H} = (H \cap K_0^0)/D$. Then $\bar{K}_0$ acts transitively and effectively on $S(V_0)$ and $\bar{K}_0/\bar{H}$ is diffeomorphic to $S(V_0)$. Put $\bar{N}_0 = N(\bar{H}, \bar{K}_0)/\bar{H}$ which is isomorphic to $N(H \cap K_0^0, K_0^0)/(H \cap K_0^0)$ as a Lie group. The pair $(\bar{K}_0, \bar{N}_0)$ is one of pairs $(G, N(H)/H)$ in the list of Lemma 4.3. Now we shall prove that $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$. If $\bar{N}_0 = Z_2$, this is clear since $\hat{c}$ is a trivial map.

Consider the case $\bar{K}_0 = SU(n)$ $(n \geq 1)$ and $\bar{N}_0 = U(1)$. In this case V_0 is an n-dimensional complex vector space and $\bar{N}_0 = U(1)$ acts on V_0 as a scalar multiplication. We can regard C^n as a 2n-dimensional real vector space R^{2n} and $\bar{N}_0$ as $SO(2)$. Then there exist smooth functions c_i: $J \longrightarrow R$, $i = 1, 2$, such that

$$\hat{c}(r) = \begin{bmatrix} c_1(r) & -c_2(r) \\ c_2(r) & c_1(r) \end{bmatrix} \in SO(2) \quad \text{for } r \in J.$$

Note that $\tilde{h}_2$: $[-\varepsilon, \varepsilon] \longrightarrow D_\varepsilon(V_0)$ is a smooth map and $\tilde{h}_2(r) = c(r) \cdot re_0 = \hat{c}(r) \cdot re_0$ for $r \neq 0$. In this case $e_0 = (1, 0, \ldots, 0) \in S^{2n-1}$ and $\tilde{h}_2(r) = (c_1(r)r, c_2(r)r, 0, \ldots, 0)$ for $r \in J$. Put $c_i(0) = \lim_{r \to 0} c_i(r)$ for $i = 1, 2$. From Lemma 4.4, $c_i: [-\varepsilon, \varepsilon] \longrightarrow R$, $i = 1, 2$, are smooth functions and $\hat{c}$ can be extended to a smooth maps on $[-\varepsilon, \varepsilon]$.

Now consider the case $\bar{K}_0 = Sp(n)$ $(n \geq 1)$ and $\bar{N} = Sp(1)$. In this case V_0 is an n-dimensional quaternionic vector space H^n and $\bar{N}_0 = Sp(1)$ acts on V_0 as a scalar multiplication ~~on the right~~. We can regard H^n as R^{4n} and $Sp(1)$ as a subgroup of $SO(4)$ naturally.
By the similar way as in the case $K_0 = SU(n)$, there exist smooth functions c_i: $J \longrightarrow R$, $i = 1, 2, 3, 4$, such that $h_2(r) = (c_1(r)r, c_2(r)r, c_3(r)r, c_4(r)r, 0, \ldots, 0)$ for $r \in J$, and we can extend $\hat{c}$ to a smooth map on $[-\varepsilon, \varepsilon]$.

The proof of the other cases are similar to those of the above cases. Thus we can extend c to a smooth map on $[-\varepsilon, \varepsilon]$. Since $c(r) \in N(H, K_0)/H$

for $r \neq 0$, we see $c(0) \in N(H,K_0)/H$. Put $\bar{a}(0) = b(0)\cdot c(0)$. Since $b(0) \in N(H) \cap N(K_0)$ and $c(0) \in N(H,K_0)/H$, we have $\bar{a}(0) \in (N(H) \cap N(K_0))/H$. Let $\hat{a}\colon [-1/2,1/2] \longrightarrow N(H)/H$ be a map given by $\hat{a}(r) = \bar{a}(r^2)$. Since $\hat{a}(r) = b(r)\cdot c(r)$ for $-\varepsilon \le r \le \varepsilon$, $\hat{a}$ is a smooth map. Since $\hat{a}$ is an even map and $\bar{a}(r) = \hat{a}(\sqrt{r})$ for $0 \le r \le 1/4$, $\bar{a}$ is a smooth map on $[0,1/4]$ by Lemma 2.4. Thus we can extend the map a to a smooth map $\bar{a}$ on $[0,1)$ satisfying $\bar{a}(0) \in (N(H) \cap N(K_0))/H$. Similarly we can extend a to a smooth map $\bar{a}$ on $[0,1]$ satisfying $\bar{a}(1) \in (N(H) \cap N(K_1))/H$. This completes the proof of Proposition 4.1.

Let Q denote the set of smooth maps $f\colon [0,1] \longrightarrow N(H)/H$ satisfying $f(i) \in (N(H) \cap N(K_i))/H$ for $i = 0,1$, endowed with C^∞ topology. Using Proposition 4.1, we define a map $L\colon \operatorname{Ker} P \longrightarrow Q$ by $L(h) = \bar{a}^{-1}$.

Lemma 4.5. $L\colon \operatorname{Ker} P \longrightarrow Q$ is a group monomorphism.

Proof. Let $h_i \in \operatorname{Ker} P$ for $i = 1,2$. For $0<r<1$ and $g \in G$, we have

$$\begin{aligned}(g\cdot L(h_2 \circ h_1)(r)^{-1}, r) &= (\alpha \circ h_2 \circ h_1 \circ \alpha^{-1})(gH, r) \\ &= ((\alpha \circ h_2 \circ \alpha^{-1}) \circ (\alpha \circ h_1 \circ \alpha^{-1}))(gH, r) \\ &= (\alpha \circ h_2 \circ \alpha^{-1})(g\cdot L(h_1)(r)^{-1}, r) \\ &= (g\cdot L(h_1)(r)^{-1} \cdot L(h_2)(r)^{-1}, r).\end{aligned}$$

Thus $L(h_2 \circ h_1) = L(h_2)\cdot L(h_1)$ on $(0,1)$. Since $L(h_1)$, $L(h_2)$ and $L(h_1 \circ h_2)$ are smooth maps on $[0,1]$, $L(h_2 \circ h_1) = L(h_2)\cdot L(h_1)$ on $[0,1]$. Thus L is a group homomorphism. Suppose $L(h) = 1$ for $h \in \operatorname{Ker} P$. Then $(h \circ \alpha^{-1})(gH, r) = \alpha^{-1}(gH, r)$ for $g \in G$ and $0<r<1$, and $h = 1$ on $\pi^{-1}((0,1))$. Thus $h = 1$ on M, and L is a monomorphism.

Propostion 4.6. L is a continuous map.

Proof. We shall use the notations in the proof of Proposition 4.1. Since L is a group homomorphism, it is sufficient to show ~~Proposition 4.6~~ that $L\colon O \longrightarrow Q$ is continuous. Let I denote a closed interval $[0,\varepsilon^2]$, $[\varepsilon^2/2, 1-\varepsilon^2/2]$ or $[1-\varepsilon^2, 1]$. By Lemma 2.3, it is sufficient to prove that $L_I\colon O \xrightarrow{L} Q \xrightarrow{j^*} C^\infty(I, N(H)/H)$ is continuous, where $j\colon I \hookrightarrow [0,1]$ is an inclusion map.

First we shall consider the case $I = [0,\varepsilon^2]$. Let L_1 be the composition

$$O \xrightarrow{(k\circ\phi_{0,1}^{-1}\circ i)^*} C^\infty([-\varepsilon,\varepsilon], p_{0,\varepsilon}^{-1}(U_0)) \xrightarrow{(\sigma_0\circ \rho_1\circ\phi_{0,1})_*} C^\infty([-\varepsilon,\varepsilon], G),$$

where $k\colon p_{0,\varepsilon}^{-1}(\bar{W}_0) \hookrightarrow M$ is an inclusion map. Then L_1 is continuous by Lemma 2.2. Note that $L_1(h) = b$.

Let $L_2\colon O \longrightarrow C^\infty([-\varepsilon,\varepsilon], (N(H,K_0)/H)^0)$ be a map given by $L_2(h) = c$. We shall prove that L_2 is continuous. This is trivial in the case $N(H,K_0)/H = Z_2$. Consider the case $\bar{K}_0 = SU(n)$ $(n \geq 2)$. In this case $V_0 = C^n = R^{2n}$ and $\bar{N}_0 = U(1) = SO(2)$. Put $C_0^\infty([-\varepsilon,\varepsilon], V_0) = \{F \in C^\infty([-\varepsilon,\varepsilon], V_0);\ F(0) = 0\}$, endowed with C^∞ topology. Let Φ: $C_0^\infty([-\varepsilon,\varepsilon], V_0) \longrightarrow C^\infty([-\varepsilon,\varepsilon], R^2)$ be a map defined by $\Phi(F) = (\Phi(F^1), \Phi(F^2))$, where $F = (F^1,\ldots,F^{2n})$ and $\Phi(F^i)$ is a map defined in Lemma 4.4. Then Φ is continuous by Lemma 4.4. Let $m\colon R^2 \longrightarrow M_2(R)$ denote a smooth map defined by

$$m(x,y) = \begin{bmatrix} x & -y \\ y & x \end{bmatrix},$$

where $M_2(R)$ denote the set of all 2×2 matrices over R. Let L_2' denote the composition

$$O \xrightarrow{(k\circ\phi_{0,1}^{-1}\circ i)^*} C^\infty([-\varepsilon,\varepsilon], p_{0,1}^{-1}(U_0)) \xrightarrow{(\rho_2\circ\phi_{0,1})_*} C^\infty([-\varepsilon,\varepsilon], D_\varepsilon(V_0)).$$

From Lemma 2.2, L_2' is continuous. Note that $L_2'(h) = \tilde{h}_2$ and $L_2'(O)$ is contained in $C_0^\infty([-\varepsilon,\varepsilon], V_0)$. Let $\hat{L}_2$ denote the composition

$$O \xrightarrow{L_2'} C_0^\infty([-\varepsilon,\varepsilon], V_0) \xrightarrow{\Phi} C^\infty([-\varepsilon,\varepsilon], R^2) \xrightarrow{m_*} C^\infty([-\varepsilon,\varepsilon], M_2(R)).$$

Then $\hat{L}_2(h) = \hat{c}$ and $\hat{L}_2$ is continuous. This implies that L_2 is continuous by using Lemma 2.2. Similarly we can wee that L_2 is continuous in the other cases.

Let $\mu\colon G \times G/H \longrightarrow G/H$ be a map defined by the left translation and let $\iota\colon (N(H,K_0)/H)^0 \hookrightarrow G/H$ be an inclusion map. Then the composition

$$\begin{aligned} \hat{L}\colon O &\xrightarrow{(L_1,\,\iota_* \circ L_2)} C^\infty([-\varepsilon,\varepsilon],\ G) \times C^\infty([-\varepsilon,\varepsilon],\ G/H) \\ &\xrightarrow{\kappa} C^\infty([-\varepsilon,\varepsilon],\ G \times G/H) \\ &\xrightarrow{\mu_*} C^\infty([-\varepsilon,\varepsilon],\ G/H) \end{aligned}$$

is continuous by Lemma 2.2, where κ is defined by $\kappa(f_1,f_2)(r) = (f_1(r),f_2(r))$. Note that $\hat{L}(h) = b \cdot c = \hat{a}$ and $\hat{L}(O)$ is contained in $C_e^\infty([-\varepsilon,\varepsilon],\ N(H)/H)$. Here $C_e^\infty([-\varepsilon,\varepsilon],\ N(H)/H)$ denote the set of all smooth even maps $f\colon [-\varepsilon,\varepsilon] \longrightarrow N(H)/H$, endowed with C^∞ topology. Let $T\colon C_e^\infty([-\varepsilon,\varepsilon],\ N(H)/H) \longrightarrow C^\infty([0,\varepsilon^2],\ N(H)/H)$ be a map defined by $T(f)(r) = f(\sqrt{r})$. By the same argument as in the proof in Lemma 2.4, we can prove that T is continuous. Thus $L_I = T \circ L$ is continuous.

Now consider the case $I = [\varepsilon^2/2,\ 1-\varepsilon^2/2]$. L_I is given by the composition

$$\begin{aligned} O &\xrightarrow{k^*} C^\infty(\pi^{-1}(I),\ \pi^{-1}(I)) \\ &\xrightarrow{(\alpha^{-1} \circ \iota)^*} C^\infty(I,\ \pi^{-1}(I)) \\ &\xrightarrow{(q_2 \circ \alpha)_*} C^\infty(I,\ G/H), \end{aligned}$$

where $k\colon \pi^{-1}(I) \hookrightarrow M$ is an inclusion, $\iota\colon I \longrightarrow G/H \times I$ is a map given by $\iota(r) = (1H,r)$ and $q_2\colon G/H \times I \longrightarrow G/H$ is the projection on the first factor. Thus L_I is continuous. We can see that L_I is continuous in the case $I = [1-\varepsilon^2,1]$ similarly as in the case $I = [0,\varepsilon^2]$, and this completes the proof of Proposition 4.6.

§5. Subgroups of the topological groups Q and $\mathrm{Ker}\,P$.

In this section we shall consider subgroups Q_1 and S of the topological groups Q and $\mathrm{Ker}\,P$, respectively, such that $L(S) = Q_1$, and we shall prove that the inclusions $Q_1 \hookrightarrow Q_0$ and $S \hookrightarrow \mathrm{Ker}\,P$ are homotopy equivalences, where Q_0 is the identity component of Q.

Put $Q_1 = \{a \in Q_0;\ a(r) = a(0) \text{ for } 0 \le r \le 1/4 \text{ and } a(r) = a(1) \text{ for } 3/4 \le r \le 1\}$. Then Q_1 is a topological subgroup of Q_0. Let $i\colon Q_1 \hookrightarrow Q_0$ be an inclusion.

Lemma 5.1. $i\colon Q_1 \hookrightarrow Q_0$ is a homotopy equivalence.

Proof. Let $\sigma\colon [0,1] \longrightarrow [0,1]$ be a smooth map such that

$$\sigma(r) = 0 \quad \text{for } 0 \le r \le 1/4,$$
$$\sigma(r) = 1 \quad \text{for } 3/4 \le r \le 1.$$

Let $\mu_t\colon [0,1] \longrightarrow [0,1]$ $(0 \le t \le 1)$ be a smooth homotopy given by $\mu_t(r) = t\sigma(r) + (1-t)r$. Since $(a \circ \mu_t)(i) \in (N(H) \cap N(K_i))/H$ for $i = 0,1$, $a \circ \mu_t$ is an element of Q. Define $q\colon Q_0 \times [0,1] \longrightarrow Q$ by $q(a,t) = a \circ \mu_t$. Let $\mu\colon [0,1] \longrightarrow C^\infty([0,1],[0,1])$ be a map given by $\mu(t) = \mu_t$. Then it is easy to see that μ is continuous. Note that q is given by the composition

$$Q_0 \times [0,1] \xrightarrow{(1,\mu)} Q_0 \times C^\infty([0,1],[0,1]) \xrightarrow{\text{comp}} C^\infty([0,1],\ N(H)/H),$$

where comp is given by $\mathrm{comp}(a,f) = a \circ f$. By Lemma 2.2 (6), q is continuous. Then $q(Q_0 \times [0,1])$ is contained in Q_0. Let $q_t\colon Q_0 \longrightarrow Q_0$ be a map given by $q_t(a) = q(a,t)$. Since $\mu_1 = \sigma$, $q_1(Q_0)$ is contained in Q_1. Thus q is a homotopy between $q_0 = 1_{Q_0}$ and $q_1 = i \circ q_1$. Note that $q_t(Q_1)$ is contained in Q_1 for any t. Then $q\colon Q_1 \times [0,1] \longrightarrow Q_1$ is a homotopy between 1_{Q_1} and $q_1 \circ i$. Therefore Lemma 5.1 follows.

Put $S = L^{-1}(Q_1) \subset \mathrm{Ker}\,P$. Let $\iota\colon S \hookrightarrow \mathrm{Ker}\,P$ be an inclusion.

Lemma 5.2. $\iota: S \hookrightarrow \mathrm{Ker}\, P$ is a homotopy equivalence.

Proof. Put $a = L(h^{-1})$ for $h \in \mathrm{Ker}\, P$. Let $h_t: M \longrightarrow M$ $(0 \le t \le 1)$ be a map as follows: h_t is given on $\pi^{-1}((0,1))$ by the composition $\pi^{-1}((0,1)) \xrightarrow{\alpha} G/H \times (0,1) \xrightarrow{\hat{h}_t} G/H \times (0,1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0,1))$, where $\hat{h}_t$ is defined by $\hat{h}_t(gH, r) = (g \cdot q_t(a)(r), r)$. $h_t(gK_i) = ga(i) \cdot K_i$ $(i = 0,1)$ for $g \in G$. Here we need the following

Assertion 5.3. h_t is a smooth map for any t.

Proof. By the definition, h_t is smooth on $\pi^{-1}((0,1))$. We shall prove that h_t is smooth on $\pi^{-1}(0)$. Let a_0 be an element of G such that $a_0 H = a(0)$ and $a_0 \in N(H) \cap N(K_0)$. For $[g,0] \in p_{0,1/2}^{-1}(1K_0)$, $(p_{0,1/2} \circ h)([g,0]) = \pi_0(ga_0) = \pi_0(a_0) \in a_0 U_0$. Then there exists a neighborhood W_0 of $1K_0$ in G/K_0 such that $(p_{0,1/2} \circ h)(p_{0,1/2}^{-1}(\bar{W}_0))$ is contained in $a_0 U_0$. For $[g, re_0] \in p_{0,1/2}^{-1}(\bar{W}_0)$ and $0 \le t \le 1$,

$$\begin{aligned}(p_{0,1/2} \circ h_t)([g, re_0]) &= \bar{\pi}_0(gq_t(a)(r^2)) \\ &= \bar{\pi}_0(ga((1-t)r^2)) \\ &= (p_{0,1/2} \circ h)([g, \sqrt{1-t}\, re_0])\end{aligned}$$

which is contained in $(p_{0,1/2} \circ h)(p_{0,1/2}^{-1}(\bar{W}_0)) \subset a_0 U_0$. Then $h_t(p_{0,1/2}^{-1}(g\bar{W}_0))$ is contained in $p_{0,1/2}^{-1}(ga_0 U_0)$ for $g \in G$ and $0 \le t \le 1$.

Let $\tilde{h}: W_0 \times D_{1/2}(V_0) \longrightarrow U_0 \times D_{1/2}(V_0)$ be a map given by $\tilde{h} = \phi_{0,ga_0} \circ h \circ \phi_{0,g}^{-1}$ for $g \in G$. Let $\rho_1: U_0 \times D_{1/2}(V_0) \longrightarrow U_0$ and $\rho_2: U_0 \times D_{1/2}(V_0) \longrightarrow D_{1/2}(V_0)$ be the projections on the first factor and the second factor respectively. Put $g' = ga_0$ and put $\tilde{h}^i = \rho_i \circ \tilde{h}$ for $i = 0,1$. Then $\tilde{h}^i$ is a smooth map and

$$\tilde{h}^1(x, rke_0) = g'^{-1} g\sigma_0(x)k \cdot \bar{\pi}_0(a(r^2)),$$

$$\tilde{h}^2(x, rke_0) = \sigma_{0,g'}(g\sigma_0(x)k \cdot \bar{\pi}_0(a(r^2)))^{-1}\, g\sigma_0(x)ka(r^2) \cdot re_0$$

for $x \in W_0$ and $k \in K_0$, where $\bar{\pi}_0: G/H \longrightarrow G/K_0$ is the natural projection.

Put $\tilde{h}_t^i = \rho_i \circ \phi_{0,g'} \circ h_t \circ \phi_{0,g}^{-1}$ for $i = 0,1$. Then

$$\tilde{h}_t^1(x, rke_0) = g'^{-1} g\sigma_0(x)k \cdot \bar{\pi}_0(a(\mu_t(r^2))),$$

$$\tilde{h}_t^2(x, rke_0) = \sigma_{0,g'}(g\sigma_0(x)k\cdot\bar{\pi}_0(a(\mu_t(r^2))^{-1}\, g\sigma_0(x)ka(\mu_t(r^2))\cdot re_0$$

for $x \in W_0$ and $k \in K_0$.

Since $\sigma(r^2) = 0$ for $r \leq 1/2$, $\mu(r^2, t) = (1-t)r^2$ for $0 \leq r \leq 1/2$. Then $\tilde{h}_t^1(x,v) = \tilde{h}^1(x, \sqrt{1-t}\, v)$ for $0 \leq t \leq 1$ and $\tilde{h}_t^2(x,v) = 1/\sqrt{1-t}\, \tilde{h}^2(x, \sqrt{1-t}\, v)$ for $0 \leq t < 1$. Thus $\tilde{h}_t^1$ $(0 \leq t \leq 1)$ and $\tilde{h}_t^2$ $(0 \leq t < 1)$ are smooth maps.

By the Taylor formula (c.f. J. Dieudonné [5, Chapter VIII, (8, 14,3)]), we have

$$\tilde{h}^2(x,v) = \tilde{h}^2(x,0) + \left(\int_0^1 (D\tilde{h}^2)(x, \zeta v)\, d\zeta\right)v,$$

where $D\tilde{h}^2$ is the derivative of $\tilde{h}^2$. Since $\tilde{h}^2(x,0) = 0$,

$$\tilde{h}_t^2(x,v) = \left(\int_0^1 (D\tilde{h}^2)(x, \sqrt{1-t}\,\zeta v)\, d\zeta\right)v \quad \text{for } 0 \leq t < 1.$$

Then $\tilde{h}_1^2(x,v) = \lim_{t \to 1} \tilde{h}_t^2(x,v) = (D\tilde{h}^2)(x,0)v$, and $\tilde{h}_1^2$ is a smooth map. Therefore h_t is smooth on $\pi^{-1}(0)$ for any $0 \leq t \leq 1$. Similarly we can prove that h_t is smooth on $\pi^{-1}(1)$, and Assertion 5.3 follows.

Proof of Lemma 5.2 continued. Let $\bar{q}\colon \operatorname{Ker} P \times [0,1] \longrightarrow \operatorname{Ker} P$ be a map defined by $\bar{q}(h,t) = h_t$. By Assertion 5.3, h_t and h_t^{-1} are smooth maps, and $\bar{q}$ is a well defined map. Next we shall prove that $\bar{q}$ is continuous. Let W_i be a neighborhood of $1K_i$ in G/K_i satisfying $\bar{W}_i \subset U_i$ for $i = 0,1$. Put $O = \{h \in \operatorname{Ker} P\,;\ h(p_{i,1/2}^{-1}(\bar{W}_i)) \subset p_{i,1/2}^{-1}(U_i)$ for $i = 0,1\}$. Then O is an open neighborhood of 1_M in $\operatorname{Ker} P$. For $h \in O$, $g \in G$ and $0 \leq t \leq 1$, $h_t(p_{i,1/2}^{-1}(g\bar{W}_i))$ is contained in $p_{i,1/2}^{-1}(gU_i)$ $(i = 0,1)$. Let W_2 be an open neighborhood of $1H$ in G/H satisfying $\bar{W}_2 \subset U_2$. Let C be one of the sets $\{p_{i,1/2}^{-1}(g\bar{W}_i)$ $(i = 0,1,\ g \in G)$, $\alpha^{-1}(g\bar{W}_2 \times [1/5, 4/5])$ $(g \in G)\}$. By Lemma 2.3, it is sufficient to show that the composition $\bar{q}_C\colon O \times [0,1] \xrightarrow{\bar{q}} \operatorname{Ker} P \xrightarrow{j_C^*} C^\infty(C,M)$ is continuous for any C, where $j_C\colon C \hookrightarrow M$ is an inclusion map.

First consider the case $C = p_{0,1/2}^{-1}(g\bar{W}_0)$. Let $\nu_1\colon C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0) \times [0,1] \longrightarrow C^\infty(\bar{W}_0 \times D_{1/2}(V_0), U_0)$ be a map given by $\nu_1(f,t)(x,v) = f(x, \sqrt{1-t}\, v)$. Let $\nu_2\colon C^\infty(\bar{W}_0 \times D_{1/2}(V_0), D_{1/2}(V_0)) \times [0,1] \longrightarrow C^\infty(\bar{W}_0 \times D_{1/2}(V_0), D_{1/2}(V_0))$ be a map given by $\nu_2(f,t)(x,v) = \left(\int_0^1 (Df)(x, \sqrt{1-t}\,\zeta v)\, d\zeta\right)(v)$. It is easy to see that ν_1 and ν_2 are continuous. Note that $\bar{q}_C$ is the composition

$$O\times[0,1]\xrightarrow{(j_C{}^*,1)} C^\infty(C,\ p_{0,1/2}^{-1}(gU_0))\times[0,1]$$

$$\xrightarrow{((\phi_{0,g})_*\circ(\phi_{0,g})^*,\ 1)} C^\infty(\bar{W}_0\times D_{1/2}(V_0),\ U_0\times D_{1/2}(V_0))\times[0,1]$$

$$\xrightarrow{((\rho_1)_*,(\rho_2)_*,1)} C^\infty(\bar{W}_0\times D_{1/2}(V_0),\ U_0)\times C^\infty(\bar{W}_0\times D_{1/2}(V_0),D_{1/2}(V_0))\times[0,1]$$

$$\xrightarrow{\nu} C^\infty(\bar{W}_0\times D_{1/2}(V_0),\ U_0)\times C^\infty(\bar{W}_0\times D_{1/2}(V_0),\ D_{1/2}(V_0))$$

$$\xrightarrow{\kappa} C^\infty(\bar{W}_0\times D_{1/2}(V_0),\ U_0\times D_{1/2}(V_0))$$

$$\xrightarrow{(\phi_{0,g}^{-1})_*\circ(\phi_{0,g}^{-1})^*} C^\infty(C,\ p_{0,1/2}^{-1}(gU_0))\hookrightarrow C^\infty(C,M).$$

Here ν is given by $\nu(f_1,f_2,t)=(\nu_1(f_1,t),\ \nu_2(f_2,t))$ and κ is the map defined in Lemma 2.2 (5). Then $\bar{q}_C$ is continuous by Lemma 2.2.

Next consider the case $C=\alpha^{-1}(g\bar{W}_2\times[1/5,4/5])$. Let $m\colon N(H)/H\times G/H\longrightarrow G/H$ be a map defined by $m(nH,gH)=(gn)H$ and $p_2\colon G/H\times[1/5,4/5]\longrightarrow[0,1]$ be a map given by $p_2(gH,r)=r$. Let $\delta\colon Q_0\longrightarrow Q_0$ be a map given by $\delta(a)=a^{-1}$. Then the map $\bar{q}_C$ is the composition

$$O\times[0,1]\xrightarrow{(L,1)} Q_0\times[0,1]\xrightarrow{\delta\circ q} Q_0\xrightarrow{p_2{}^*} C^\infty(G/H\times[1/5,4/5],\ N(H)/H)$$

$$\xrightarrow{(1_{G/H\times[1/5,4/5]})!} C^\infty(G/H\times[1/5,4/5],\ N(H)/H\times G/H\times[1/5,4/5])$$

$$\xrightarrow{m_*} C^\infty(G/H\times[1/5,4/5],\ G/H\times[1/5,4/5])$$

$$\xrightarrow{(\alpha\circ j_C)^*\circ(\alpha^{-1})_*} C^\infty(C,\ \alpha^{-1}(G/H\times[1/5,4/5])\hookrightarrow C^\infty(C,M),$$

which is continuous because L and q are continuous.

Similarly as in the case $C=p_{0,1/2}^{-1}(g\bar{W}_0)$, we can see that $\bar{q}_C$ is continuous in the case $C=p_{1,1/2}^{-1}(g\bar{W}_1)$. Thus $\bar{q}$ is continuous. Since $q_1(Q_0)\subset Q_1$, $\bar{q}_1(\mathrm{Ker}\,P)\subset S$. Therefore $\bar{q}$ is a homotopoy between $\bar{q}_0=1_{\mathrm{Ker}\,P}$ and $\bar{q}_1=\iota\circ\bar{q}_1$. Since $q(Q_1\times[0,1])\subset Q_1$, $\bar{q}(S\times[0,1])\subset S$. Then $\bar{q}\colon S\times[0,1]\longrightarrow S$ is a homotopy between 1_S and $\bar{q}_1\circ\iota$. Thus ι is a homotopy equivalence, and this completes the proof of Lemma 5.2.

§6. Proof of Theorem.

In this section, we shall see that $L: S \longrightarrow Q_1$ is an isomorphism between topological groups, and we shall prove our Theorem.

Proposition 6.1. $L: S \longrightarrow Q_1$ is an isomorphism between topological groups.

Before the proof of Propositin 6.1, we begin with some lemmas. For any topological subgroup K of G, K^0 denotes the identity component of K.

Lemma 6.2. For any $a \in N(K_0)^0 \cap N(H)$, there exist $a' \in N(H^0) \cap K_0^0$ and $n \in \mathrm{Cent}(K_0^0)$ such that $a = n \cdot a'$, where $\mathrm{Cent}(K_0^0)$ is the centralizer of K_0^0 in G.

Proof. Since $N(K_0)^0$ is a compact connected Lie group, there exist a torus group T and a simply connected semi-simple compact Lie group G' such that $\hat{N}_0 = T \times G'$ is a finite covering group of $N(K_0)^0$ (c.f. L. Pontrjagin [8, §64]). Let $q_0: \hat{N}_0 \longrightarrow N(K_0)^0$ be the covering projection. Put $\hat{K}_0 = q_0^{-1}(K_0^0)$. Since K_0^0 is a normal subgroup of $N(K_0)^0$, $\hat{K}_0$ is a normal subgroup of $\hat{N}_0$. Then $\hat{K}_0^0$ is also a normal subgroup of $\hat{N}_0$. Here we need the following :

Assertion 6.3. There exists a closed normal subgroup K_0' of $\hat{N}_0$ such that $\hat{N}_0$ is isomorphic to the product group $\hat{K}_0^0 \times K_0'$ as a Lie group.

Proof. There exist simply connected simple Lie groups G_i $(1 \le i \le r)$ such that $G' = G_1 \times \dots \times G_r$. Since $\hat{K}_0^0$ is a compact connected Lie group, there exist simply connected simple Lie groups K_j $(1 \le j \le s)$ and a torus group T' such that $\tilde{K}_0 = T' \times K_1 \times \dots \times K_s$ is a finite covering of $\hat{K}_0^0$. Let $p_0: \tilde{K}_0 \longrightarrow \hat{K}_0^0$ be the covering projection. Let $\rho_i: \hat{N}_0 = T \times G_1 \times \dots \times G_r \longrightarrow G_i$ be a projection on the direct factor G_i $(1 \le i \le r)$. Since $\hat{K}_0^0$ is a normal subgroup of $\hat{N}_0$, $\rho_i(\hat{K}_0^0)$ is a normal subgroup of G_i.

Since G_i is a simple Lie group, $\rho_i(\hat{K}_0^0) = G_i$ or $\{1\}$. If $\rho_i(\hat{K}_0^0) = G_i$, $\rho_i(p_0(K_j))$ is a normal subgroup of G_i. Thus $\rho_i(p_0(K_j)) = G_i$ or $\{1\}$, for $1 \le i \le r$, $1 \le j \le s$.

Put $\rho_i' = \rho_i \circ p_0$. If $\rho_i'(K_{j_1}) = \rho_i'(K_{j_2})$ $(j_1 \neq j_2)$, then $\rho_i'(g_1) \cdot \rho_i'(g_2)$ $= \rho_i'(g_1 \cdot g_2) = \rho_i'(g_2 \cdot g_1) = \rho_i'(g_2) \cdot \rho_i'(g_1)$ for $g_1 \in K_{j_1}$, $g_2 \in K_{j_2}$. Then $\rho_i'(K_{j_1})$ is a commutative normal subgroup of G_i, and $\rho_i'(K_{j_1}) = \{1\}$. If $\rho_i'(K_j) = G_i$, then $\rho_i'(T')$ is a normal subgroup of G_i, hence $\rho_i'(T')$ $= \{1\}$. Therefore, if $\rho_i'(K_j) = G_i$, then $\rho_i'(T') = \{1\}$ and $\rho_i'(K_n) = \{1\}$ for $n \neq j$.

Assume $\rho_{i_1}'(K_j) = G_{i_1}$ and $\rho_{i_2}'(K_j) = G_{i_2}$ for $i_1 \neq i_2$. Let $\rho' : \tilde{K}_0 \longrightarrow G_{i_1} \times G_{i_2}$ be a map defined by $\rho'(k) = (\rho_{i_1}'(k), \rho_{i_2}'(k))$. Since $\hat{K}_0^0$ is a normal subgroup of $\hat{N}_0$ and $\rho'(\tilde{K}_0) = \rho'(K_j)$, $\rho'(K_j)$ is a normal subgroup of $G_{i_1} \times G_{i_2}$. Then, for $x, y \in K_j$, there exists $k \in K_j$ such that $(\rho_{i_1}'(x), 1)\rho'(y)(\rho_{i_1}'(x)^{-1}, 1) = \rho'(k)$. Then $\rho_{i_1}'(xyx^{-1}) =$ $\rho_{i_1}'(x)\rho_{i_1}'(y)\rho_{i_1}'(x)^{-1} = \rho_{i_1}'(k)$ and $\rho_{i_2}'(y) = \rho_{i_2}'(k)$. Since K_j, G_{i_n} $(n = 1, 2)$ are simply connected simple Lie groups, $\rho_{i_n}' : K_j \to G_{i_n}$ is an isomorphism between the Lie groups. Thus $xyx^{-1} = k = y$ for any $x, y \in K_j$, and K_j must be a commutative Lie group, which is a contradiction since K_j is a simple Lie group.

Thus we may assume that $\rho_j'(K_j) = G_j$ and $\rho_i'(K_j) = \{1\}$ $(i \neq j)$ for $1 \le j \le s$, $1 \le i \le r$. For $i > s$, $\rho_i(\hat{K}_0^0) = \rho_i'(\tilde{K}_0) = \rho_i'(T')$ which is a commutative normal subgroup of G_i, hence $\rho_i'(T') = \{1\}$. Then $p_0(T')$ is a subgroup of T, and there exists a torus subgroup S of T such that $T = p_0(T') \times S$. Put $K' = S \times G_{s+1} \times \cdots \times G_r$. Then $\hat{N}_0 = \hat{K}_0^0 \times K_0'$, and Assertion 6.3 follows.

Proof of Lemma 6.2 continued. By Assertion 6.3, there exists a closed normal subgroup K_0' of $\hat{N}_0$ such that $\hat{N}_0 = \hat{K}_0^0 \times K_0'$. Since K_0^0 is a connected group, $q_0(\hat{K}_0^0) = K_0^0$. Then $N(K_0)^0 = q_0(\hat{N}_0) = q_0(\hat{K}_0^0) \cdot q_0(K_0')$

$= K_0^0 \cdot q_0(K_0')$. Note that $q_0(K_0')$ is contained in $\mathrm{Cent}(K_0^0)$. Thus, for $a \in N(K_0)^0 \cap N(H)$, there exists $a' \in K_0^0$ and $n \in \mathrm{Cent}(K_0^0)$ such that $a = a' \cdot n$. Since $N(H) \subset N(H^0)$ and $H^0 \subset K_0^0$, $H^0 = aH^0a^{-1} = a'nH^0n^{-1}a'^{-1}$ $= a'H^0a'^{-1}$. Thus $a' \in N(H^0)$ and Lemma 6.2 follows.

For $a \in Q_1$, we define a map $h\colon M \to M$ as follows:

$$h(\alpha^{-1}(gH,r)) = \alpha^{-1}((ga(r)^{-1},r)) \quad \text{for } (gH,r) \in G/H \times (0,1),$$
$$h([g,0]) = [ga(i)^{-1},0] \quad \text{for } [g,0] \in \pi^{-1}(i) \ (i = 0,1).$$

Lemma 6.4. h is a smooth map.

Proof. Choose $a_0 \in (N(H) \cap N(K_0))^0 \subset N(H)^0 \cap N(K_0)^0$ such that $a(0)^{-1} = a_0H$. There exists a neighborhood W_0 of $1K_0$ in G/K_0 such that $\pi_0^{-1}(W_0) \cdot a_0$ is contained in $a_0 \cdot \pi_0^{-1}(U_0)$. Since $a(r) = a(0)$ for $0 \le r \le 1/4$, $h(p_{0,1/2}^{-1}(gW_0))$ is contained in $p_{0,1/2}^{-1}(ga_0U_0)$. Let $\tilde{h}_1\colon W_0 \times D_{1/2}(V_0) \longrightarrow U_0$ be a map given by the composition $\rho_1 \circ \phi_{0,ga_0} \circ h \circ \phi_{0,g}^{-1}$, and let $\tilde{h}_2\colon W_0 \times D_{1/2}(V_0) \longrightarrow D_{1/2}(V_0)$ be a map given by the composition $\rho_2 \circ \phi_{0,ga_0} \circ h \circ \phi_{0,g}^{-1}$. Note that

$$\begin{aligned}(h \circ \phi_{0,g}^{-1})((x,rke_0)) &= h([g\sigma_0(x)k, re_0]) \\ &= h(\alpha^{-1}((g\sigma_0(x)kH, r^2))) \\ &= \alpha^{-1}((g\sigma_0(x)ka_0H, r^2)) \\ &= [g\sigma_0(x)ka_0, re_0]\end{aligned}$$

for $x \in W_0$, $k \in K_0$, $0 < r \le 1/2$. Since $a_0 \in N(K_0)$, $ka_0K_0 = a_0K_0$. Then

$$\tilde{h}_1(x,v) = a_0^{-1}\sigma_0(x)a_0K_0 \quad \text{for } (x,v) \in W_0 \times D_{1/2}(V_0), \text{ and}$$
$$\tilde{h}_2(x,rke_0) = \sigma_{0,ga_0}(g\sigma_0(x)a_0K_0)^{-1}g\sigma_0(x)ka_0 \cdot re_0 \quad \text{for}$$

$x \in W_0$, $k \in K_0$, $0 \le r \le 1/2$. Thus $\tilde{h}_1$ is a smooth map and $\tilde{h}_2$ is smooth on $W_0 \times (D_{1/2}(V_0) - 0)$. We shall prove that $\tilde{h}_2$ is smooth on $W_0 \times 0$, hence h is smooth on $\pi^{-1}(0)$. This is trivial in the case $\dim S(V_0) = 0$.

Let $\xi_{a_0,g}\colon W_0 \to G$ be a map given by $\xi_{a_0,g}(x) = \sigma_{0,ga_0}(g\sigma_0(x)a_0K_0)^{-1}g\sigma_0(x)$. Then $\xi_{a_0,g}$ is a smooth map. By Lemma 6.2, there exist $a_0' \in N(H^0) \cap K_0^0$ and $n \in \mathrm{Cent}(K_0^0)$ such that $a_0 = na_0'$. Then $\widetilde{h}_2(x, rke_0) = \xi_{a_0,g}(x)kna_0' \cdot rke_0 = \xi_{a_0,g}(x)nka_0' \cdot re_0$ for $x \in W_0$, $k \in K_0^0$ and $0 \le r \le 1/2$. Note that $N(H^0) \cap K_0^0 = N(H^0, K_0^0)$.

Assertion 6.5. For $a \in N(H^0, K_0^0)$, let $\psi_a : D(V_0) \longrightarrow D(V_0)$ be a map defined by $\psi_a(rke_0) = rkae_0$ for $0 \le r \le 1$, $k \in K$. Then ψ_a is a diffeomorphism. Moreover, let $\psi : N(H^0, K_0^0) \longrightarrow \mathrm{Diff}^\infty(D(V_0))$ be a map given by $\psi(a) = \psi_a$, then ψ is continuous.

Proof. If $\dim S(V_0) = 0$, then $K_0^0 \subset H$ and $\psi_a = 1_{D(V_0)}$. In this case, the proof is trivial. We assume $\dim S(V_0) > 0$. Since $S(V_0) = K_0/H$ is connected, K_0^0 acts transitively on $S(V_0)$. Let L be the ineffective kernel of the action $K_0^0 \times S(V_0) \to S(V_0)$. Put $\bar{K} = K_0^0/L$ and $\bar{H} = (H \cap K_0^0)/L$. Then $\bar{K}$ acts transitively and effectively on $S(V_0)$ and $\bar{H}$ is an isotropy subgroup of this action. By Lemma 4.3, $\bar{K}$, $\bar{H}$ and $N(\bar{H}, \bar{K})/\bar{H}$ are G, H and $N(H)/H$ in Lemma 4.3, respectively. Hence $\bar{H}$ is connected. Since the identity component of $H \cap K_0^0$ is H^0, $\bar{H} = H^0 \cdot L/L$. For $a \in N(H^0, K_0^0)$, the left coset aL is an element of $N(\bar{H}, \bar{K})$. Then a defines an element $\tilde{a} \in N(\bar{H}, \bar{K})/\bar{H}$. Note that $\psi_a(rke_0) = rkae_0 = rk\tilde{a}e_0$ for $0 \le r \le 1$, $k \in K_0^0$.

Consider the case $\bar{K} = SU(n)$ $(n \ge 2)$, $\bar{H} = SU(n-1)$ and $N(\bar{H}, \bar{K})/\bar{H} = U(1)$. In this case, $V_0 = C^n$ and $U(1)$ acts on V_0 as a scalar multiplication. Thus $\psi_a(rke_0) = \tilde{a} \cdot rke_0$ for $rke_0 \in D(V_0)$. Hence ψ_a is a diffeomorphism. It is easy to see that ψ is continuous.

Next consider the case $\bar{K} = Sp(n)$ $(n \ge 1)$, $\bar{H} = Sp(n-1)$ and $N(\bar{H}, \bar{K})/\bar{H} = Sp(1)$. In this case, $V_0 = H^n$ and $Sp(1)$ acts on V_0 as a scalar multiplication on the right. Then $\psi_a(v) = v \cdot \tilde{a}$ for $v \in D(V_0)$, hence ψ_a is a diffeomorphism and ψ is continuous. Similarly we can see that ψ_a is a diffeomorphism and ψ is continuous in the other cases, and Assertion 6.5 follows.

Proof of Lemma 6.4 continued. Since $\widetilde{h}_2(x, v) = \xi_{a_0,g}(x)n \cdot \psi_{a_0'}(v)$, by Assertion 6.5, $\tilde{h}_2$ is a smooth map. Thus $\tilde{h}_1$ and $\tilde{h}_2$ are smooth maps, hence h is smooth on $\pi^{-1}(0)$. Similarly we can see that h is smooth on $\pi^{-1}(1)$. By the definition, h is smooth on $\pi^{-1}((0,1))$, and this completes the proof of Lemma 6.4.

Let $\hat{L}(a)$ be a smooth map $h: M \longrightarrow M$ in Lemma 6.4, for $a \in Q_1$. Since $\hat{L}(a^{-1}) = \hat{L}(a)^{-1}$, h is a diffeomorphism of M. By the definition, h is an equivariant map. Thus we have a map $\hat{L}: Q_1 \longrightarrow \mathrm{Diff}_G^\infty(M)$. Note that $\hat{L}$ is an abstract group homomorphism.

Lemma 6.6. $\hat{L}: Q_1 \longrightarrow \mathrm{Diff}_G^\infty(M)$ is continuous.

Proof. Let W_i be a neighborhood of $1K_i$ in G/K_i such that $\bar{W}_i \subset U_i$ $(i = 0,1)$, and let W_2 be a neighborhood of 1H in G/H such that $\bar{W}_2 \subset U_2$. Put $A_i = \{n \in N(K_i)^0;\ n^{-1}\bar{W}_i n \subset U_i\}$. Then A_i is an open neighborhood of the idetity in $N(K_i)^0$. Let $q_i: \hat{N}_i \longrightarrow N(K_i)^0$ be a finite covering such that $\hat{N}_i$ is a direct product $T_i \times G_i'$. Here T_i is a torus group and G_i' is a simply connected semi-simple compact Lie group. Put $\hat{K}_i = q_i^{-1}(K_i^0)$. By Assertion 6.3, there exists a closed normal subgroup K_i' of $\hat{N}_i$ such that $\hat{N}_i = \hat{K}_i^0 \times K_i'$. Let s_i be a smooth local cross section of q_i defined on an open neighborhood B_i of the identity in $N(K_i)^0$. Since $\pi_2: (N(H) \cap N(K_i))^0 \longrightarrow ((N(H) \cap N(K_i))/H)^0$ is a fibration, there exists a smooth local cross section t_i of π_2 defined on an open neighborhood E_i of 1H such that $t_i(E_i) \subset A_i \cap B_i$.

Put $O = \{a \in Q_1;\ a(i)^{-1} \in E_i\ (i = 0,1)\}$. Then O is an open neighborhood of the identity. Since $\hat{L}$ is a group homomorphism, it is enough to show ~~Lemma 6.6~~ that $\hat{L}$ is continuous on O. Let C denote one of the sets $\{p_{i,1/2}^{-1}(g\bar{W}_i)\ (i = 0,1,\ g \in G),\ \alpha^{-1}(g\bar{W}_2 \times [1/5,4/5])\ (g \in G)\}$. By Lemma 2.3, if $\hat{L}_C: O \xrightarrow{\hat{L}} \mathrm{Diff}_G^\infty(M)^0 \xrightarrow{j_C^*} C^\infty(C,M)$ is continuous for any C, then $\hat{L}$ is continuous, where $j_C: C \hookrightarrow M$ is an inclusion map.

First consider the case $C = p_{0,1/2}^{-1}(g\bar{W}_i)$. Let $\beta_1 : \hat{N}_0 = \hat{K}_0^0 \times K_0' \longrightarrow \hat{K}_0^0$ and $\beta_2 : \hat{N}_0 \longrightarrow K_0'$ be the projection on the first factor and the second factor respectively. Let L_1 be the composition

$$O \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \cap B_0 \xrightarrow{(\xi_g,\ q_0 \circ \beta_2 \circ s_0)} C^\infty(\bar{W}_0, G) \times \mathrm{Cent}(K_0^0) \xrightarrow{m} C^\infty(\bar{W}_0, G).$$

Here r, ξ and m are given by $r(a) = a(0)^{-1}$, $\xi_g(a_0)(x) = \xi_{a_0,g}(x)$ and $m(f,n)(x) = f(x)\cdot n$, respectively. Put $a_0 = (t_0 \circ r)(a)$ for $a \in O$. Then $\pi_0(\xi_{g,a_0}(x)) = \pi_0(a_0^{-1})$ for $w \in \bar{W}_0$ and $\pi_0((q_0 \circ \beta_2 \circ s_0)(a_0)) = \pi_0(a_0)$. Therefore $L_1(a) \in K_0$ for any $a \in O$, and $L_1(O) \subset C^\infty(\bar{W}_0, K_0)$. Let L_2 be the composition

$$O \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \cap B_0 \xrightarrow{q_0 \circ \beta_1 \circ s_0} N(H^0, K_0^0) \xrightarrow{\psi} \mathrm{Diff}^\infty(D_{1/2}(V_0)).$$

By Assertion 6.5, L_2 is continuous. Let L_3 be the composition

$$\begin{aligned} O &\xrightarrow{(L_1, L_2)} C^\infty(\bar{W}_0, K_0) \times \mathrm{Diff}^\infty(D_{1/2}(V_0)) \\ &\xrightarrow{(\rho_1^*,\ \rho_2^*)} C^\infty(\bar{W}_0 \times D_{1/2}(V_0),\ K_0) \times C^\infty(\bar{W}_0 \times D_{1/2}(V_0),\ D_{1/2}(V_0)) \\ &\xrightarrow{\kappa} C^\infty(\bar{W}_0 \times D_{1/2}(V_0),\ K_0 \times D_{1/2}(V_0)) \\ &\xrightarrow{\mu_*} C^\infty(\bar{W}_0 \times D_{1/2}(V_0),\ D_{1/2}(V_0)), \end{aligned}$$

where μ is given by $\mu(k,v) = k\cdot v$, and κ is the map in Lemma 2.2. Then L_3 is continuous, and $L_3(a) = \tilde{h}_2$. Let $\gamma : A_0 \longrightarrow C^\infty(\bar{W}_0, U_0)$ be a map defined by $\gamma(a_0)(x) = a_0^{-1}\sigma_0(x) a_0 K_0$. γ is a restriction map to A_0 of a map $\bar{\gamma} : N(K_0) \longrightarrow C^\infty(G/K_0, G/K_0)$ given by $\bar{\gamma}(n)(gK_0) = n^{-1} g n K_0$. Since $\bar{\gamma}$ is a continuous map, γ is continuous. Let L_4 be the composition

$$O \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \xrightarrow{\gamma} C^\infty(\bar{W}_0, U_0) \xrightarrow{\rho_1^*} C^\infty(\bar{W}_0 \times D_{1/2},\ U_0).$$

Then L_4 is continuous and $L_4(h) = \tilde{h}_1$. L_C is the composition

$$\begin{aligned} O &\xrightarrow{(L_4, L_3)} C^\infty(\bar{W}_0 \times D_{1/2}(V_0),\ U_0) \times C^\infty(\bar{W}_0 \times D_{1/2}(V_0),\ D_{1/2}(V_0)) \\ &\xrightarrow{\kappa} C^\infty(\bar{W}_0 \times D_{1/2}(V_0),\ U_0 \times D_{1/2}(V_0)) \\ &\xrightarrow{(\phi_{0,g})^*\ (\phi_{0,g})_*} C^\infty(C,\ p_{0,1/2}^{-1}(gU_0)) \hookrightarrow C^\infty(C, M). \end{aligned}$$

Thus L_C is continuous.

Now consider the case $C = \alpha^{-1}(g\bar{W}_2 \times [1/5, 4/5])$. Let $m : g\bar{W}_2 \times N(H)/H$

$\longrightarrow G/H$ be a map defined by $m(gH, nH) = gnH$, and let $\rho\colon G/H \times [1/5, 4/5] \longrightarrow [1/5, 4/5]$ be the projection on the second factor. Then $\hat{L}_C$ is given by the composition

$$
\begin{aligned}
O &\xrightarrow{i^{*}\circ\delta_{*}} C^{\infty}([1/5,4/5],\ N(H)/H) \\
&\xrightarrow{(1_{g\bar{W}_2}\,)_!} C^{\infty}(g\bar{W}_2\times[1/5,4/5],\ g\bar{W}_2\times N(H)/H) \\
&\xrightarrow{m_*} C^{\infty}(g\bar{W}_2\times[1/5,4/5],\ G/H) \\
&\xrightarrow{\rho_{\#}} C^{\infty}(g\bar{W}_2\times[1/5,4/5],\ G/H\times[1/5,4/5]) \\
&\xrightarrow{\alpha^{*}\circ(\alpha^{-1})_{*}} C^{\infty}(C,\ \alpha^{-1}(G/H\times[1/5,4/5])) \hookrightarrow C^{\infty}(C,M),
\end{aligned}
$$

where $i\colon [1/5,4/5] \hookrightarrow [0,1]$ is the inclusion map and $\delta\colon N(H)/H \longrightarrow N(H)/H$ is a map given by $\delta(a) = a^{-1}$. By Lemma 2.2, $\hat{L}_C$ is continuous.

We can see that L_C is continuous in the case $C = p_{1,1/2}^{-1}(g\bar{W}_1)$ similarly as in the case $C = p_{0,1/2}^{-1}(g\bar{W}_0)$, and this completes the proof of Lemma 6.6.

Proof of Proposition 6.1. From Lemma 6.6, $\hat{L}(Q_1)$ is contained in $\mathrm{Diff}_G^{\infty}(M)_0$. Then, by the definition, $\hat{L}(Q_1)$ is contained in S, and $\hat{L} = L^{-1}$. Combining Lemma 4.5, Proposition 4.6 and Lemma 6.6, $\hat{L}\colon S \longrightarrow Q_1$ is an isomorphism between topological groups, and this completes the proof of Proposition 6.1.

Proof of Theorem. By Corollary 3.6, $\mathrm{Diff}_G^{\infty}(M)_0$ has the same homotopy type as $\mathrm{Ker}\,P$. Combining Lemma 5.1, Lemma 5.2 and Proposition 6.1, $\mathrm{Ker}\,P$ has the same homotopy type as Q_0. Note that Q_0 has the same homotopy type as the path space $\Omega(N(H)/H;\ (N(H)\cap N(K_0))/H,\ (N(H)\cap N(K_1))/H)_0$. This completes the proof of our Theorem.

§7. Concluding remarks.

From our Theorem, we have the following:

Corollary 7.1. (1) If $K_0 = K_1 = G$, then $\mathrm{Diff}_G^\infty(M)_0$ has the same homotopy type as $(N(H)/H)^0$.

(2) If $N(H)/H$ is a finite group, then $\mathrm{Diff}_G^\infty(M)_0$ is cotractible.

Remark 7.2. In K. Abe and K. Fukui [1], we have proved that $\mathrm{Diff}_G^\infty(M)_0$ is perfect if M is a G-manifold with one orbit type and $\dim M/G \geq 1$. But, by using Proposition 3.1, we can see that $\mathrm{Diff}_G^\infty(M)_0$ is not perfect in the case $M/G = [0,1]$.

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